The $\top, \bot$ approach for truncated semantics

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Abstract. In previous work [1] we argue that in order for temporal logic to be useful in incomplete verification methods such as simulation or bounded model checking, it is necessary to define semantics over truncated paths. A truncated path is a path which is finite, but not necessarily maximal. The semantics on truncated paths is based on three views: the weak view, the neutral view and the strong view.
In [1] we have defined the semantics explicitly for each view. That is, the semantics of formulas is interpreted over a pair consisting of a *word* over 2^P (where P is the set of atomic propositions) and a *view* which is either *weak* or *neutral* or *strong*. Here we suggest an alternative, more concise, approach for giving the semantics, which is achieved by interpreting the formulas over words from the alphabet $2^P \cup \{\top, \bot\}$ where $\top$ and $\bot$ are special symbols.

1 Introduction

Traditional LTL semantics [2] are defined for maximal paths in the model. A maximal path is either an infinite path or a path whose last state has no successor in the model. In a previous work [1] we have defined LTL over *truncated paths*. A truncated path is a path that is finite, but not necessarily maximal. For instance, a path examined by a simulation tool, or by bounded model checking, is a truncated path. Truncated paths are also convenient for reasoning about hardware resets because a reset can be thought of as "cutting" a maximal path into two disjoint parts – a finite, truncated part up until the reset and a maximal (possibly infinite) part after the reset.

Methods of reasoning about finite maximal paths are insufficient for reasoning about truncated paths. When considering a truncated path, the user might want to reason about properties of the truncation as well as properties of the model. For instance, the user might want to specify that a simulation test goes on long enough to discharge all outstanding obligations, or, on the other hand, that an obligation need not be met if it "is the fault of the test" (that is, if the test is too short). The former approach is useful for a test designed (either manually or by other means) to continue until correct output can be confirmed. The latter approach is useful for a test which has no "opinion" on the correct length of a test - for instance, a monitor running concurrently with the main test to check for bus protocol errors.

In such a situation, we need to define the semantics over a truncated path. In other words, at the end of the truncated path, the truth value must be decided. If the path was truncated after the evaluation of the formula completed, the truth value is already determined. The problem is to decide the truth value if the path was truncated before the evaluation of the formula completed, i.e., where there is *doubt* regarding what would have been the truth value if the path had not been truncated. We term a decision to return *true* when there is doubt the *weak view* and a decision to return *false* when there is doubt the *strong view.* For instance, consider the formula $p \rightarrow \mathsf{X}\ q$ on a finite path of length 1 such that p holds in the first state. Given the evidence we have, it is impossible to say whether or not the formula holds neutrally on the untruncated path. Thus the formula holds weakly and does not hold strongly.

The formula $p \rightarrow \mathsf{X}\ q$ holds weakly on a finite path of length 1 such that p holds in the first state. In fact, on such a path the formula $p \rightarrow \mathsf{X}\ \varphi$ holds weakly for any φ, including $\varphi = \mathit{false}$. Thus, the weak view of the truncated semantics takes the lazy approach, in that it is not required to know whether or not φ is satisfiable if we have not yet reached a point "where φ matters".

In [1] we have defined the semantics explicitly for each view. That is, the semantics of formulas is interpreted over a pair consisting of a *word* over 2^P (where P is the set of atomic propositions) and a *view* which is either *weak* or *strong* or *neutral* (corresponding to satisfaction on maximal paths). Here we suggest an alternative, more concise, approach for giving the semantics, which is achieved by interpreting the formulas over words from the alphabet $2^P \cup \{\top, \bot\}$ where $\top$ and $\bot$ are special symbols. This approach and the equivalence of the two semantics are the topics of this paper.

2 Preliminaries

2.1 Notation

Throughout, i, j, k, and n denote non-negative integers.

We will denote a letter from some alphabet Σ by ℓ (possibly with subscripts) and an empty, finite, or infinite word over Σ by u, v, or w (possibly with subscripts). The concatenation of u and v is denoted by

uv. If u is infinite, then $uv = u$. The empty word is denoted by ϵ, so that $w\epsilon = \epsilon w = w$. If $w = uv$, we say that u is a *prefix* of w, denoted $u \preceq w$, that v is a *suffix* of w, and that w is an *extension* of u, denoted $w \succeq u$. We use $u \prec w$ to mean $u \preceq w$ and $u \neq w$.

We denote the length of word w as $|w|$. The empty word ϵ has length 0, a finite word $w = (\ell_0\ell_1\ell_2 \cdots \ell_n)$ has length $n + 1$, and an infinite word has length ω.[1] For $i < |w|$, we use w^i to denote the $i + 1^{st}$ letter of w (since counting of letters starts at zero), and we denote by $w^{i..}$ the suffix of w starting at index i. When $i \leq j < |w|$, we denote by $w^{i..j}$ the finite sequence of letters starting from index i and ending in index j. That is, $w^{i..j} = (w^i w^{i+1} \cdots w^j)$. We use ℓ^ω to denote an infinite-length word, each letter of which is ℓ.

We use the term *boolean expression* to mean an element of the free boolean algebra $\mathbf{B} = 2^{2^P}$. For a letter $\ell \in 2^P$ and a boolean expression $b \in \mathbf{B}$, we say that ℓ *satisfies* b, denoted $\ell \Vdash b$, iff $\ell \in b$. We write *true* to denote the element $2^P \in \mathbf{B}$ and *false* to denote the element $\emptyset \in \mathbf{B}$. For any letter $\ell \in 2^P$, $\ell \Vdash$ *true* and $\ell \nVdash$ *false*.

2.2 The logic

In [1] we defined the logic LTL^{trunc}, extending LTL[2] with truncation operators trunc_W and trunc_S.

Definition 1 (LTL^{trunc} **formulas**).

- *Every boolean expression is an* LTL^{trunc} *formula.*
- *If φ and ψ are* LTL^{trunc} *formulas and b is a boolean expression, then the following are* LTL^{trunc} *formulas*[2]*:*
  - $\neg\varphi$ • $\varphi \wedge \psi$ • $\mathsf{X!}\ \varphi$ • $[\varphi\ \mathsf{U}\ \psi]$ • φ trunc_W b

Additional operators are defined as syntactic sugaring of the above operators:

- $\varphi \vee \psi \stackrel{\text{def}}{=} \neg(\neg\varphi \wedge \neg\psi)$
- $\mathsf{X}\ \varphi \stackrel{\text{def}}{=} \neg(\mathsf{X!}\ \neg\varphi)$
- $\mathsf{F}\ \varphi \stackrel{\text{def}}{=} [\mathit{true}\ \mathsf{U}\ \varphi]$
- $\mathsf{G}\ \varphi \stackrel{\text{def}}{=} \neg\mathsf{F}\ \neg\varphi$
- $[\varphi\ \mathsf{W}\ \psi] \stackrel{\text{def}}{=} [\varphi\ \mathsf{U}\ \psi] \vee \mathsf{G}\varphi$
- φ trunc_S $b \stackrel{\text{def}}{=} \neg(\neg\varphi$ trunc_W $b)$

[1] We use ω to denote the cardinality of the non-negative integers. It is understood that $\omega + \omega = \omega$ and $\omega + n = n + \omega = \omega$.

[2] Since we have finite paths, we need both weak and strong next-time operators. We use X! for the strong version. The weak version, X, is given as syntactic sugar.

2.3 The three-view approach to the truncated semantics

The *truncated semantics* of LTL^{trunc} was defined in [1]. Later, we will give an alternative formalization of the same semantics. In order to distinguish between the two, we will call the formalization given in [1] the *three-view approach* to the truncated semantics. The three-view approach defines the semantics of an LTL^{trunc} formula with respect to finite or infinite words over 2^P and a context indicating the *view*, which can be either weak, neutral, or strong. According to our motivation presented above, the formula φ holds on a truncated path in the weak view if up to the point where the path ends, "nothing has yet gone wrong" with φ. It holds on a truncated path in the neutral view according to the standard LTL semantics for finite paths. In the strong view, φ holds on a truncated path if everything that needs to happen to convince us that φ holds on the original untruncated path has already occurred. Intuitively then, each view is recursively defined, with negation switching between the weak and strong views. The truncation operators trunc_w and trunc_s truncate a path and move to the weak and strong views, respectively.

We use $w \models^{S} \varphi$ to denote that φ is satisfied under the model (w, S), where S is "$-$" if the view is weak, null if it is neutral, and "$+$" if it is strong. Under the neutral view only non-empty words are evaluated; under the weak/strong views, empty words are evaluated as well. The definition makes use of an "overflow" and "underflow" for the indices of a word w. That is, $w^{j..} = \epsilon$ if $j \geq |w|$, and $w^{i..h} = \epsilon$ if $h < i < |w|$ (h possibly negative). For example, in the definition below of the semantics of $[\varphi \mathsf{U} \psi]$ under the weak and strong views, when we say "$\exists k$", k is not required to be less than $|w|$.

Let φ and ψ denote LTL^{trunc} formulas. In the three-view approach, the truncated semantics is defined as follows.[3]

holds weakly: For w such that $|w| \geq 0$,

1. $w \models^{-} b \iff |w| = 0$ or $w^0 \Vdash b$
2. $w \models^{-} \neg\varphi \iff w \not\models^{+} \varphi$
3. $w \models^{-} \varphi \wedge \psi \iff w \models^{-} \varphi$ and $w \models^{-} \psi$
4. $w \models^{-} \mathsf{X!}\ \varphi \iff w^{1..} \models^{-} \varphi$
5. $w \models^{-} [\varphi \mathsf{U} \psi] \iff \exists k$ such that $w^{k..} \models^{-} \psi$, and for every $j < k$, $w^{j..} \models^{-} \varphi$
6. $w \models^{-} \varphi$ trunc_w $b \iff w \models^{-} \varphi$ or $\exists k < |w|$ s.t. $w^k \Vdash b$ and $w^{0..k-1} \models^{-} \varphi$

holds neutrally: For w such that $|w| > 0$,

[3] Recall that b always denotes a boolean expression and i, j, k, n always denote non-negative integers.

1. $w \models b \iff w^0 \Vdash b$
2. $w \models \neg\varphi \iff w \not\models \varphi$
3. $w \models \varphi \wedge \psi \iff w \models \varphi$ and $w \models \psi$
4. $w \models \mathsf{X!}\ \varphi \iff |w| > 1$ and $w^{1..} \models \varphi$
5. $w \models [\varphi \mathsf{U} \psi] \iff \exists k < |w|$ such that $w^{k..} \models \psi$, and for every $j < k$, $w^{j..} \models \varphi$
6. $w \models \varphi$ $\mathsf{trunc\_w}$ $b \iff w \models \varphi$ or $\exists k < |w|$ s.t. $w^k \Vdash b$ and $w^{0..k-1} \models^- \varphi$

holds strongly: For w such that $|w| \geq 0$,

1. $w \models^+ b \iff |w| > 0$ and $w^0 \Vdash b$
2. $w \models^+ \neg\varphi \iff w \not\models^- \varphi$
3. $w \models^+ \varphi \wedge \psi \iff w \models^+ \varphi$ and $w \models^+ \psi$
4. $w \models^+ \mathsf{X!}\ \varphi \iff w^{1..} \models^+ \varphi$
5. $w \models^+ [\varphi \mathsf{U} \psi] \iff \exists k$ such that $w^{k..} \models^+ \psi$, and for every $j < k$, $w^{j..} \models^+ \varphi$
6. $w \models^+ \varphi$ $\mathsf{trunc\_w}$ $b \iff w \models^+ \varphi$ or $\exists k < |w|$ s.t. $w^k \Vdash b$ and $w^{0..k-1} \models^- \varphi$

In [1], we analyzed the characteristics of the truncated semantics. We showed that the strong view is indeed stronger than the neutral, and the neutral stronger than the weak (the Strength Relation Theorem). We also showed that if a truncated path w satisfies φ in the weak view, then any prefix of w satisfies φ in the weak view, and that if a truncated path w satisfies φ in the strong view, then any extension of w satisfies φ in the strong view (the Prefix/Extension Theorem). Finally, we defined the definitive prefix of a word w with respect to φ ($dp(w, \varphi)$) as the shortest prefix of w which suffices to conclude that φ holds or does not hold on w, and showed that any proper prefix of $dp(w, \varphi)$ satisfies weakly both φ and $\neg\varphi$, while $dp(w, \varphi)$ and all of its extensions satisfy strongly exactly one of φ or $\neg\varphi$ (the Definitive Prefix Theorem).

3 The $\top, \bot$ approach to the truncated semantics

The idea behind the $\top, \bot$ approach is that we add special letters $\top$ and $\bot$ such that $\top$ satisfies all Boolean expressions, including *false*, and $\bot$ satisfies no Boolean expression, including *true*. We would like the recursive definition to achieve that $\top^\omega$ satisfies all LTL^{trunc} formulas, and $\bot^\omega$ satisfies no LTL^{trunc} formula. We define negation to switch a $\top$ to a $\bot$ and vice versa. And we define the semantics of $\mathsf{trunc\_w}$ so that it pads the word with $\top^\omega$ and the semantics of $\mathsf{trunc\_s}$ so that it pads the word with $\bot^\omega$. Intuitively, this will enforce the requirement that the weak view return *true* and the strong view *false* when there is doubt as to the validity of the formula on the original, untruncated path.

Formally, the $\top,\bot$ approach to the truncated semantics defines the semantics of LTL^{trunc} with respect to non-empty finite and infinite words over $\Sigma = 2^P \cup \{\top,\bot\}$. We use $w \models_{\top,\bot} \varphi$ to denote that φ is satisfied on non-empty word w in the $\top,\bot$ approach. We use $\overline{w}$ to denote the *dual* of w, the word obtained by replacing every $\top$ with a $\bot$ and vice versa. We augment the boolean relation $\Vdash$ of Section 2.1 as follows to include the two special letters $\top$ and $\bot$: for every b, $\top \Vdash b$ and $\bot \not\Vdash b$. Note that in particular, $\top \Vdash$ *false* and $\bot \not\Vdash$ *true*. We make use of the "underflow" for the indices of w, so that $w^{i..h} = \epsilon$ if $h < i < |w|$ (h possibly negative).

Let φ and ψ denote LTL^{trunc} formulas. In the $\top,\bot$ approach, the truncated semantics is defined as follows.

1. $w \models_{\top,\bot} b \iff w^0 \Vdash b$
2. $w \models_{\top,\bot} \neg\varphi \iff \overline{w} \not\models_{\top,\bot} \varphi$
3. $w \models_{\top,\bot} \varphi \wedge \psi \iff w \models_{\top,\bot} \varphi$ and $w \models_{\top,\bot} \psi$
4. $w \models_{\top,\bot} \mathsf{X!}\ \varphi \iff |w| > 1$ and $w^{1..} \models_{\top,\bot} \varphi$
5. $w \models_{\top,\bot} [\varphi \mathsf{U} \psi] \iff \exists k < |w|$ s.t. $w^{k..} \models_{\top,\bot} \psi$, and $\forall j < k,\ w^{j..} \models_{\top,\bot} \varphi$
6. $w \models_{\top,\bot} \varphi\ \mathsf{trunc\_W}\ b \iff w \models_{\top,\bot} \varphi$ or $\exists k < |w|$ s.t. $w^k \Vdash b$ and $w^{0..k-1}\top^\omega \models_{\top,\bot} \varphi$

For any word w over Σ, if $w\top^\omega \models_{\top,\bot} \varphi$ we say that w *weakly satisfies* φ, denoted $w \models^-_{\top,\bot} \varphi$. If $w\bot^\omega \models_{\top,\bot} \varphi$ we say that w *strongly satisfies* φ, denoted $w \models^+_{\top,\bot} \varphi$. For infinite w, $w\top^\omega = w\bot^\omega = w$ (by the definition of concatenation in Section 2.1), so $w \models^-_{\top,\bot} \varphi$ iff $w \models_{\top,\bot} \varphi$ iff $w \models^+_{\top,\bot} \varphi$.

The following theorem states the equivalence of the two approaches to the truncated semantics.

Theorem 2 (Equivalence of three-view and $\top,\bot$ approaches). *Let u be a word over 2^P, v be a non-empty word over 2^P, and φ be a formula of* LTL^{trunc}*. Then*

1. $u \models^- \varphi \iff u \models^-_{\top,\bot} \varphi$
2. $v \models \varphi \iff v \models_{\top,\bot} \varphi$
3. $u \models^+ \varphi \iff u \models^+_{\top,\bot} \varphi$

Proof. Let w be any word over 2^P, and let φ be a formula of LTL^{trunc}. We will show that

a. $w \models^- \varphi \iff w\top^\omega \models_{\top,\bot} \varphi$
b. $w \models^+ \varphi \iff w\bot^\omega \models_{\top,\bot} \varphi$

The proof of the second item, i.e. that for a non-empty word v, $v \models \varphi \iff v \models_{\top\!,\bot} \varphi$, follows by inspection of the definitions, since v is over 2^P and we have the equivalence (a) from above to use in the case of the semantics of trunc_w.

The proof proceeds by induction over the structure of φ.

1. $\varphi = b$.
   (a) $w \models^{-} b$
   $\iff |w| = 0$ or $w^0 \Vdash b$
   $\iff (w\top^\omega)^0 = \top$ or $(w\top^\omega)^0 \Vdash b$
   $\iff (w\top^\omega)^0 \Vdash b$
   $\iff w\top^\omega \models_{\top\!,\bot} b$
   (b) $w \models^{+} b$
   $\iff |w| > 0$ and $w^0 \Vdash b$
   $\iff (w\bot^\omega)^0 \Vdash b$
   $\iff w\bot^\omega \models_{\top\!,\bot} b$
2. $\varphi = \neg\psi$.
   (a) $w \models^{-} \neg\psi$
   $\iff w \not\models^{+} \psi$
   $\iff$ [induction] $w\bot^\omega \not\models_{\top\!,\bot} \psi$
   $\iff [\overline{w} = w]\ (\overline{w}\bot^\omega) \not\models_{\top\!,\bot} \psi$
   $\iff w\top^\omega \models_{\top\!,\bot} \neg\psi$
   (b) $w \models^{+} \neg\psi$
   $\iff w \not\models^{-} \psi$
   $\iff$ [induction] $w\top^\omega \not\models_{\top\!,\bot} \psi$
   $\iff [\overline{w} = w]\ (\overline{w}\top^\omega) \not\models_{\top\!,\bot} \psi$
   $\iff w\bot^\omega \models_{\top\!,\bot} \neg\psi$
3. $\varphi = \psi \wedge \vartheta$.
   (a) $w \models^{-} \psi \wedge \vartheta$
   $\iff w \models^{-} \psi$ and $w \models^{-} \vartheta$
   $\iff$ [induction] $w\top^\omega \models_{\top\!,\bot} \psi$ and $w\top^\omega \models_{\top\!,\bot} \vartheta$
   $\iff w\top^\omega \models_{\top\!,\bot} \psi \wedge \vartheta$
   (b) $w \models^{+} \psi \wedge \vartheta$
   $\iff w \models^{+} \psi$ and $w \models^{+} \vartheta$
   $\iff$ [induction] $w\bot^\omega \models_{\top\!,\bot} \psi$ and $w\bot^\omega \models_{\top\!,\bot} \vartheta$
   $\iff w\bot^\omega \models_{\top\!,\bot} \psi \wedge \vartheta$
4. $\varphi = \mathsf{X!}\ \psi$.
   (a) $w \models^{-} \mathsf{X!}\ \psi$
   $\iff w^{1..} \models^{-} \psi$
   $\iff$ [induction] $w^{1..}\top^\omega \models_{\top\!,\bot} \psi$

$\iff (w\top^\omega)^{1..} \models_{\top,\bot} \psi$
$\iff [|w\top^\omega| = \omega]\ w\top^\omega \models_{\top,\bot} \mathsf{X!}\ \psi$

(b) $w \models^{+} \mathsf{X!}\ \psi$
$\iff w^{1..} \models^{+} \psi$
$\iff$ [induction] $w^{1..}\bot^\omega \models_{\top,\bot} \psi$
$\iff (w\bot^\omega)^{1..} \models_{\top,\bot} \psi$
$\iff [|w\bot^\omega| = \omega]\ w\bot^\omega \models_{\top,\bot} \mathsf{X!}\ \psi$

5. $\varphi = [\psi\ \mathsf{U}\ \vartheta]$.

(a) $w \models^{-} [\psi\ \mathsf{U}\ \vartheta]$
$\iff$ there exists k such that $w^{k..} \models^{-} \vartheta$ and for all $j < k, w^{j..} \models^{-} \psi$
$\iff$ [induction] there exists k such that $w^{k..}\top^\omega \models_{\top,\bot} \vartheta$ and for all $j < k, w^{j..}\top^\omega \models_{\top,\bot} \psi$
$\iff$ there exists k such that $(w\top^\omega)^{k..} \models_{\top,\bot} \vartheta$ and for all $j < k, (w\top^\omega)^{j..} \models_{\top,\bot} \psi$
$\iff w\top^\omega \models_{\top,\bot} [\psi\ \mathsf{U}\ \vartheta]$

(b) $w \models^{+} [\psi\ \mathsf{U}\ \vartheta]$
$\iff$ there exists k such that $w^{k..} \models^{+} \vartheta$ and for all $j < k, w^{j..} \models^{+} \psi$
$\iff$ [induction] there exists k such that $w^{k..}\bot^\omega \models_{\top,\bot} \vartheta$ and for all $j < k, w^{j..}\bot^\omega \models_{\top,\bot} \psi$
$\iff$ there exists k such that $(w\bot^\omega)^{k..} \models_{\top,\bot} \vartheta$ and for all $j < k, (w\bot^\omega)^{j..} \models_{\top,\bot} \psi$
$\iff w\bot^\omega \models_{\top,\bot} [\psi\ \mathsf{U}\ \vartheta]$

6. $\varphi = \psi$ $\mathsf{trunc\_w}$ b.

(a) $w \models^{-} \psi$ $\mathsf{trunc\_w}$ b
$\iff$ either $w \models^{-} \psi$ or there exists $k < |w|$ such that $w^k \Vdash b$ and $w^{0..k-1} \models^{-} \psi$
$\iff$ [induction]
A: either $w\top^\omega \models_{\top,\bot} \psi$ or there exists $k < |w|$ such that $w^k \Vdash b$ and $w^{0..k-1}\top^\omega \models_{\top,\bot} \psi$
$\implies [|w\top^\omega| = \omega, k < |w|$ implies $(w\top^\omega)^k = w^k$ and $(w\top^\omega)^{0..k-1} = w^{0..k-1}]$
B: either $w\top^\omega \models_{\top,\bot} \psi$ or there exists $k < |w\top^\omega|$ such that $(w\top^\omega)^k \Vdash b$ and $(w\top^\omega)^{0..k-1}\top^\omega \models_{\top,\bot} \psi$
$\iff w\top^\omega \models_{\top,\bot} \psi$ $\mathsf{trunc\_w}$ b.

It remains to show that B $\implies$ A. Suppose there exists $k < |w\top^\omega|$ such that $(w\top^\omega)^k \Vdash b$ and $(w\top^\omega)^{0..k-1}\top^\omega \models_{\top,\bot} \psi$. If $k < |w|$, then $(w\top^\omega)^k = w^k$ and $(w\top\omega)^{0..k-1} = w^{0..k-1}$, so we are done. Otherwise, w is finite and $k \geq |w|$, so $w\top^\omega = (w\top^\omega)^{0..k-1}\top^\omega \models_{\top,\bot} \psi$, and again we are done.

(b) $w \models^{+} \psi$ $\mathsf{trunc\_w}$ b
$\iff$ either $w \models^{+} \psi$ or there exists $k < |w|$ such that $w^k \Vdash b$ and $w^{0..k-1} \models^{-} \psi$
$\iff$ [induction]
A: either $w\bot^\omega \models_{\top,\bot} \psi$ or there exists $k < |w|$ such that $w^k \Vdash b$ and $w^{0..k-1}\top^\omega \models_{\top,\bot} \psi$

$\Longrightarrow$ $[|w\bot^\omega| = \omega, k < |w|$ implies $(w\bot^\omega)^k = w^k$ and $(w\bot^\omega)^{0..k-1} = w^{0..k-1}]$
B: either $w\bot^\omega \models_{\top,\bot} \psi$ or there exists $k < |w\bot^\omega|$ such that $(w\bot^\omega)^k \Vdash b$ and $(w\bot^\omega)^{0..k-1}\top^\omega \models_{\top,\bot} \psi$

$\Longleftrightarrow$ $w\bot^\omega \models_{\top,\bot} \psi$ $\mathsf{trunc\_w}$ b.

It remains to show that B $\Longrightarrow$ A. Suppose there exists $k < |w\bot^\omega|$ such that $(w\bot^\omega)^k \Vdash b$ and $(w\bot^\omega)^{0..k-1}\top^\omega \models_{\top,\bot} \psi$. If w is finite and $k \geq |w|$, then $\bot = (w\bot^\omega)^k \Vdash b$, a contradiction. Therefore, $k < |w|$. Then $(w\bot^\omega)^k = w^k$ and $(w\bot^\omega)^{0..k-1} = w^{0..k-1}$, so we are done.

□

The following lemma states that indeed we get that $\top^\omega$ satisfies every formula whereas $\bot^\omega$ satisfies none.

Lemma 1. *Let φ be an* LTL^{trunc} *formula. Then*

1. $\top^\omega \models_{\top,\bot} \varphi$
2. $\bot^\omega \not\models_{\top,\bot} \varphi$

Proof. The proof follows from Theorem 2 together with Lemma 4 from [1] stating that $\epsilon \models^- \varphi$ and $\epsilon \not\models^+ \varphi$ for any LTL^{trunc} formula φ.

4 Conclusions

We have shown that the three-views semantics of LTL over truncated (and untruncated) words as defined in [1] can be given more concisely as presented here by enhancing the alphabet with two special symbols ($\top$ and $\bot$).

References

1. C. Eisner, D. Fisman, J. Havlicek, Y. Lustig, A. McIsaac, and D. Van Campenhout. Reasoning with temporal logic on truncated paths. In *Proc. 15th International Conference on Computer Aided Verification (CAV)*, LNCS 2725, pages 27–39, July 2003.
2. A. Pnueli. A temporal logic of concurrent programs. *Theoretical Computer Science*, 13:45–60, 1981.